Symmetry breaking in low-dimensional $SU(N)$ antiferromagnets

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Consequences of explicit symmetry breaking in a physically motivated model of $SU(N)$ antiferromagnet in spatial dimensions one and two are studied. It is shown that the case $N=3$, which can be realized in spin-1 cold atom systems, displays special properties distinctly different from those for $N\geq 4$. Qualitative form of the phase diagram depending on the model parameters is given.

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I. INTRODUCTION

During the last several years, there has been a revival of interest[1–12] in unconventional types of spin ordering in systems with higher spins $S\geq 1$. This interest is to large extent motivated by experiments on Bose-Einstein condensates of cold atoms with internal spin states.[13,14] Particularly, ordering of the quadrupole degrees of freedom corresponds to the so-called "spin nematic" type of spin order,[15–17] which is difficult to obtain in conventional magnetic materials since its existence requires the presence of strong non-Heisenberg (biquadratic or multispin) exchange terms, or the presence of strong frustration mixing ferromagnetic and antiferromagnetic couplings.[18–20] In strongly frustrated systems with ferromagnetic couplings, higher multipolar orders may win over the nematic one becoming dominant correlations.[19–21] For cold spinful bosons in optical lattices, strong non-Heisenberg exchange appears in the effective spin model,[22,23] favoring spin-nematic order. In higher-spin systems, higher symmetries may naturally arise. $SU(N)$ generalizations of Heisenberg spin systems in one and two spatial dimensions have been extensively studied.[24–29] Several recent studies[30–32] explore exotic pairing possibilities opened by the existence of higher $SU(N)$ symmetries with $N>2$ in fermionic systems.

In the present paper, we will study what happens to an $SU(N)$ antiferromagnet (AF) if the high symmetry gets explicitly broken by a weak perturbation. It will be shown that, similarly to $N=2$, the physically important case $N=3$ is in many respects special and breaking the $SU(3)$ symmetry leads to rich behavior which might be realizable in cold atom setups. We will see that perturbing the $SU(3)$ symmetry has a drastic effect on the topology, which is reflected in physical properties due to the role of the Berry phases. Our starting point will be the $S=1$ model on an anisotropic square lattice described by the Hamiltonian

$$\mathcal{H}=\sum_{\boldsymbol{n}}\{\hat{h}_{\boldsymbol{n},\boldsymbol{n}+x}+\lambda\hat{h}_{\boldsymbol{n},\boldsymbol{n}+y}\},$$

$$\hat{h}_{\boldsymbol{n},\boldsymbol{n}'}=\cos\,\theta(\boldsymbol{S}_{\boldsymbol{n}}\cdot\boldsymbol{S}_{\boldsymbol{n}'})+\sin\,\theta(\boldsymbol{S}_{\boldsymbol{n}}\cdot\boldsymbol{S}_{\boldsymbol{n}'})^2, \tag{1}$$

where $\boldsymbol{S}_{\boldsymbol{n}}$ is a spin-1 operator at the lattice site $\boldsymbol{n}$ and λ is the parameter controlling anisotropy of the lattice $0<\lambda<1$. This model appears, particularly, in the physics of ultracold alkali atoms with hyperfine $S=1$ spins (e.g., ^{23}Na) in optical lattices at odd filling.[23] The parameter θ can be varied by tuning the ratio a_2/a_0 of scattering lengths in $S=2$ and 0 channels using the Feshbach resonance, as well as by creating a gradient in the optical lattice potential.[33] Similar models have been proposed[7,8] as a possible explanation for the unconventional spin state discovered recently[1] in the quasi-two-dimensional (2D) $S=1$ magnet $NiGa_2S_4$ and have been also discussed[9,10] in the context of the deconfined quantum criticality conjecture.[34] In one dimension ($d=1$), this model has been extensively studied and a number of analytical[35–41] and numerical[3–6,42] results are available. In two dimensions, it was recently studied numerically by means of quantum Monte Carlo technique[9] and analytically with the help of a field-theoretical approach.[10,43]

Using the standard representation of the $S=1$ operator $S^{\alpha}_{\boldsymbol{n}}=-i\epsilon_{\alpha\beta\gamma}t^{\dagger}_{\boldsymbol{n},\beta}t_{\boldsymbol{n},\gamma}$ through three bosonic operators t_{α}, $\alpha=1,\ldots 3$ satisfying the hardcore constraint

$$t^{\dagger}_{\alpha}t_{\alpha}=n_c=1, \tag{2}$$

one can cast the local Hamiltonian in the form

$$\hat{h}_{i,j}=-Jt^{\dagger}_{i,\alpha}t^{\dagger}_{j,\beta}t_{j,\alpha}t_{i,\beta}-\tilde{J}t^{\dagger}_{i,\alpha}t^{\dagger}_{j,\alpha}t_{j,\beta}t_{i,\beta},$$

$$J\equiv-\cos\,\theta,\quad \tilde{J}=\cos\,\theta-\sin\,\theta. \tag{3}$$

Since the model is formulated in terms of local bilinears of bosonic operators $\boldsymbol{t}_{\boldsymbol{n}}$, it obviously has the local $U(1)$ symmetry for any values of the model parameters. We will be interested in the interval $-3\pi/4<\theta<0$. It is convenient to generalize the Hamiltonian (2) and (3) by letting the boson flavor index run from 1 to N and allowing the parameter n_c in Eq. (2) to be an arbitrary integer number. In case of the related models for cold atoms in optical lattices, n_c has the meaning of the number of atoms per lattice site,[22] and in what follows we will assume n_c to be odd. For $n_c=1$, $N=3$ corresponds to Eq. (1), $N=2$ describes the spin-$\frac{1}{2}$ XXZ model with $J_x=-2(J+\tilde{J})$, $J_z=2(\tilde{J}-J)$, and $N=4$ can be realized[44] as a "bilayer" spin-$\frac{1}{2}$ model with four-spin interaction between the layers,

$$\hat{h}_{\boldsymbol{n},\boldsymbol{n}'}=(2\cos\,\theta-\sin\,\theta)[(\boldsymbol{s}_{\boldsymbol{n}}\cdot\boldsymbol{s}_{\boldsymbol{n}'})+(\boldsymbol{\tau}_{\boldsymbol{n}}\cdot\boldsymbol{\tau}_{\boldsymbol{n}'})]$$
$$+4\sin\,\theta(\boldsymbol{s}_{\boldsymbol{n}}\cdot\boldsymbol{s}_{\boldsymbol{n}'})(\boldsymbol{\tau}_{\boldsymbol{n}}\cdot\boldsymbol{\tau}_{\boldsymbol{n}'}), \tag{4}$$

which is essentially the Kugel-Khomskii spin-orbital model[45]

with spin and orbital degrees of freedom described by $\boldsymbol{s}_n$ and $\boldsymbol{\tau}_n$ spin-$\frac{1}{2}$ operators, respectively; a large number of results are available for this model in one dimension.[46–55]

The point $\theta=-\pi/2$ ($J=0$) is remarkable since it has an enhanced symmetry. The Hamiltonian (1) is always $SU(2)$ [generally, $O(N)$] invariant, but at $\theta=-\pi/2$ the symmetry group is enlarged to $SU(3)$ [respectively, $SU(N)$]: since the lattice is bipartite, a transformation $\boldsymbol{t}_n \mapsto U\boldsymbol{t}_n$ on sites $\boldsymbol{n}$ belonging to sublattice A leaves the Hamiltonian invariant if it is accompanied by a conjugate transformation $\boldsymbol{t}_n \mapsto U^*\boldsymbol{t}_n$ for $\boldsymbol{n} \in B$, with a unitary matrix U.

Our strategy will be to construct an effective-field-theoretical description of the problem, using $\theta=-\pi/2$ ($J=0$) as a starting point, and to treat the term proportional to J as a perturbation. We will also see that a rich behavior is generated by adding another perturbation, namely, the *easy-axis* single-ion anisotropy to the $S=1$ Hamiltonian (1),

$$\mathcal{H} \mapsto \mathcal{H} - D\sum_{\boldsymbol{n}} (S_{\boldsymbol{n}}^z)^2, \quad D > 0. \tag{5}$$

For the case of a general N this amounts to including the term of the form $D\Sigma_{\boldsymbol{n}} t^{\dagger}_{\boldsymbol{n},N} t_{\boldsymbol{n},N}$, which breaks the symmetry down from $SU(N)$ to $SU(N-1)$. In cold atom systems, such terms appear naturally in presence of external magnetic field due to the quadratic Zeeman effect.[23,56]

The structure of the paper is as follows: In Sec. II the effective continuum theory in the vicinity of the $SU(N)$-symmetric point is derived, Sec. III considers the influence of the θ perturbation breaking the symmetry down to $O(N)$, Sec. IV studies the effects of the anisotropy (5), and finally, Sec. V contains a brief summary.

II. EFFECTIVE-FIELD THEORY IN THE VICINITY OF THE $SU(N)$ POINT

To construct the continuum field description, consider a path-integral representation of the problem, effectively replacing the bosonic operators $t_{\boldsymbol{n},\alpha}$ with complex fields on the lattice satisfying the constraint in Eq. (2). To pass to the continuum properly, one should notice that local spin-quadrupolar correlations are of the ferromagnetic type for $J+\tilde{J}>0$, while the spin-dipolar correlations are antiferromagnetic provided $\tilde{J}>J$ (Ref. 57); this can be also seen from the numerical results (see Fig. 8 of Ref. 42). We will be interested mainly in the region of $\theta<0$, where the first of those inequalities is always satisfied, but the second one breaks for $\theta<\theta_0\approx -0.65\pi$. The theory derived here will be valid for $\theta>\theta_0$ and the proper effective theory for $\theta<\theta_0$ can be found in Ref. 58.[59]

The AF character of local spin correlations suggests the following ansatz for the bosonic lattice fields $\boldsymbol{t}_n$:

$$\boldsymbol{t}_n = (\boldsymbol{u}_n + i\,\eta_{\boldsymbol{n}}\boldsymbol{v}_n) + (\eta_{\boldsymbol{n}}\boldsymbol{\varphi}_n + i\boldsymbol{\zeta}_n), \tag{6}$$

where $\eta_{\boldsymbol{n}}$ is an oscillating factor taking value ± 1 for $\boldsymbol{n}$ belonging to A and B sublattices, respectively, and $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{\varphi}$, and $\boldsymbol{\zeta}$ are assumed to be smooth functions of the site coordinate $\boldsymbol{n}$. Defining $z_n=(\boldsymbol{u}_n+i\boldsymbol{v}_n)/\sqrt{n_c}$ and $\boldsymbol{\psi}_n=(\boldsymbol{\varphi}_n+i\boldsymbol{\zeta}_n)/\sqrt{n_c}$, one can rewrite the above ansatz in a simpler form,

$$\boldsymbol{t}_n = \sqrt{n_c} \times \begin{cases} z_n + \boldsymbol{\psi}_n, & \boldsymbol{n} \in A \\ z_n^* - \boldsymbol{\psi}_n^*, & \boldsymbol{n} \in B \end{cases}, \tag{7}$$

where the constraints

$$|z|^2 + |\boldsymbol{\psi}|^2 = 1, \quad \boldsymbol{\psi}\cdot z^* + z\cdot\boldsymbol{\psi}^* = 0 \tag{8}$$

are implied. One can expect that the magnitude of $\boldsymbol{\psi}$, which corresponds to ferromagnetic fluctuations, will be much smaller than that of z. Using the ansatz (7), passing to the continuum, retaining only up to quadratic terms in $\boldsymbol{\psi}$, and neglecting its derivatives, one readily obtains the Euclidean action $\mathcal{A}=\mathcal{A}_0+\mathcal{A}_{\text{int}}+\mathcal{A}_B$, where $\mathcal{A}_0$ corresponds to $J=0$,

$$\mathcal{A}_0 = \sqrt{\lambda} n_c^2 \int d\tau \int d^2x \Bigg\{ \frac{1}{n_c}(\boldsymbol{\psi}^*\cdot\partial_\tau z - \boldsymbol{\psi}\cdot\partial_\tau z^*) + 4\tilde{J}(1+\lambda) \times [|\boldsymbol{\psi}|^2 - |\boldsymbol{\psi}^*\cdot z|^2] + \tilde{J}[|\partial_k z|^2 - |z^*\cdot\partial_k z|^2] + \mu_1(\boldsymbol{\psi}\cdot z^* + \boldsymbol{\psi}^*\cdot z) + \mu_2(|z|^2 + |\boldsymbol{\psi}|^2 - 1) \Bigg\}, \tag{9}$$

and $\mathcal{A}_B$ is the topological Berry phase contribution

$$\mathcal{A}_B = in_c \sum_{\boldsymbol{n},\tau} \eta_{\boldsymbol{n}} \arg[z_{\boldsymbol{n}}^*(\tau)\cdot z_{\boldsymbol{n}}(\tau+d\tau)] \tag{10}$$

which is known to play a crucial role in the physics of the system.[24,25,60] It is important to realize[10,22] that the naive continuum limit of Eq. (10),

$$\mathcal{A}_B = n_c \int d\tau \sum_{\boldsymbol{n}} \eta_{\boldsymbol{n}} z_{\boldsymbol{n}}^*\cdot\partial_\tau z_{\boldsymbol{n}}, \tag{11}$$

can only capture the contributions from smooth field configurations and in case of dominant nematic correlations, when z becomes a real vector defined up to a sign, misses the additional phase stemming from disclinations.

The term $\mathcal{A}_{\text{int}}$ is determined by the "perturbation" J,

$$\mathcal{A}_{\text{int}} = J\sqrt{\lambda} n_c^2 \int d\tau \int d^2x \Bigg\{ -|z^2|^2 + |z\cdot\partial_k z|^2 + \left[z^2\boldsymbol{\psi}^{*2} + \frac{1}{2} z^2 (\partial_k z^*)^2 + \text{c.c.} \right] \Bigg\}. \tag{12}$$

Here the index k runs over two spatial coordinates, the factor $\sqrt{\lambda}$ in Eq. (9), Eq. (12) comes from rescaling one of those coordinates to compensate for the anisotropy of interactions, and $\mu_{1,2}$ are the Lagrange multipliers ensuring the constraints.

For $J=0$, one can easily integrate out $\boldsymbol{\psi}$ and μ_1 fields; it turns out that $\mu_1=-n_c^{-1} z^*\cdot\partial_\tau z$, which yields

$$\boldsymbol{\psi} = -[4\tilde{J}n_c(1+\lambda)]^{-1}\{\partial_\tau z - z(z^*\cdot\partial_\tau z)\},$$

$$\boldsymbol{\psi}^* = [4\tilde{J}n_c(1+\lambda)]^{-1}\{\partial_\tau z^* + z^*(z^*\cdot\partial_\tau z)\}. \tag{13}$$

Substituting this back into Eq. (9), one obtains the effective action for z field only, where we can now approximately assume $|z|^2=1$. Rescaling the imaginary time axis $\tau \mapsto \tau/(2n_c\tilde{J}\sqrt{1+\lambda})$, one arrives at the effective action

$$\mathcal{A}_0 = \frac{1}{2g}\int d^{d+1}x\{|\partial_\mu z|^2 - |z^*\cdot\partial_\mu z|^2\}, \quad g = \frac{\sqrt{1+\lambda^{-1}}}{n_c}, \tag{14}$$

where $d=2$ is the spatial dimension, and the index μ runs over all $d+1$ space-time coordinates. Had we started with a single $S=1$ chain instead of the square lattice, we would have obtained the action of the same form (14) but with $d=1$ and $g=1/n_c$. This is nothing but the action of the CP^{N-1} model,[61–65] originally proposed as an effective theory for $SU(N)$ antiferromagnets by Read and Sachdev.[24,25] This action has a local $U(1)$ gauge symmetry $z\mapsto e^{i\varphi(x)}z$ and can be rewritten in the form

$$\mathcal{A}_0 = \frac{1}{2g}\int d^{d+1}x|(\partial_\mu - iA_\mu)z|^2, \tag{15}$$

where $A_\mu = i(z\cdot\partial_\mu z^*)$ is the $U(1)$ gauge field.

The CP^{N-1} model without the topological phase term is always gapped in $d=1$ and displays an ordering transition in $d=2$ at a certain critical value of the coupling constant.[63,64] In the disordered phase the z field acquires a finite mass and a kinetic term for the gauge field is dynamically generated,[64]

$$\mathcal{A} \mapsto \mathcal{A} + \frac{N}{4e_0^2}\int d^{d+1}xF_{\mu\nu}^2, \tag{16}$$

where $F_{\mu\nu}=\partial_\mu A_\nu - \partial_\nu A_\mu$, and the coupling constant $e_0^2 \propto \Delta^{3-d}$.

The Berry phase term (10) is crucial for the physics of the disordered phase;[24,25,60] particularly, it leads to spontaneous dimerization in $d=1$ for odd n_c (except for $N=2$ which is special: in that case the system remains gapless and translationally invariant in a wide g range[66–69]), and in two dimensions the disordered phase gets spontaneously dimerized in different patterns depending on the value of (n_c mod 4). We will come back to the role of the Berry term later and look into the rest of the action first.

III. EFFECT OF THE $SU(N)\mapsto O(N)$ PERTURBATION

The perturbing action (12) explicitly breaks the global $SU(N)$ symmetry down to $O(N)$ but preserves the $U(1)$ gauge symmetry. Consequently, nonzero J can produce only gauge-invariant perturbation terms of the form

$$|z^2|^2, \quad |z\cdot D_\mu z|^2, \ldots,$$

where $D_\mu \equiv \partial_\mu - iA_\mu$ and the ellipsis stands for terms with higher derivatives. It is easy to see that the first term above is relevant for $d<3$, while the second one is irrelevant for $d>1$ (for $d=1$ it is marginal). It thus makes sense to consider only the effect of the most relevant term, which brings us to the perturbed action $\mathcal{A}_\gamma = \mathcal{A}_0 + \mathcal{A}_{\rm int}$, with

$$\mathcal{A}_{\rm int} = -\frac{\gamma}{2g}\int d^{d+1}x|z^2|^2, \quad \gamma \simeq \frac{J}{\tilde{J}}. \tag{17}$$

It is easy to generalize the standard large-N mean-field analysis[63] of the CP^{N-1} model to include the effect of the $SU(N)$-breaking perturbation γ. We consider the action

$$\mathcal{A}_{\rm MF} = \frac{1}{2g}\int d^{d+1}\{x|\partial_\mu z|^2 - |z^*\cdot\partial_\mu z|^2 - \gamma|z^2|^2 + \sigma(|z|^2-1)\}, \tag{18}$$

where σ is the Lagrange multiplier responsible for the constraint $|z|^2=1$ and expand it around a stationary saddle-point solution $z=z_0$ and $\sigma=\sigma_0$. This expansion has to be performed differently depending on whether the perturbation is of the "nematic" ($\gamma>0$) or "antiferromagnetic" ($\gamma<0$) type.

A. Nematic side ($\gamma>0$)

In this case the saddle point can be chosen in the form $z_0=(n_0, 0, \ldots, 0)$, which in our original $N=3$ model corresponds to the spin-nematic (quadrupolar) order. Fluctuations around the mean-field solution, $z=z_0+\boldsymbol{u}+i\boldsymbol{v}$ can be described by two real N-component vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Due to the constraint $|z|^2=1$ one can set $u_1\approx 0$ and the gauge-fixing condition (e.g., setting n_0 to be real) yields $v_1\approx 0$. After integration over quadratic fluctuations, the saddle-point equations are obtained as

$$n_0^2 + g(N-1)\sum_{\boldsymbol{k}}\left\{\frac{1}{\sigma_0+k^2} + \frac{1}{\sigma_0+4\gamma+k^2}\right\} = 1,$$

$$\sigma_0 n_0 = 0, \tag{19}$$

where the sum is over $(d+1)$-dimensional reciprocal space. In one spatial dimension $d=1$, the model is disordered ($n_0=0$) for any value of the coupling constant g, and the field z is always massive with $\sigma_0=\Delta^2$ having the meaning of the squared spectral gap,

$$\Delta^2 \simeq \Lambda^2 \exp\left\{-\frac{2\pi}{g(N-1)}\right\} - 2\gamma, \quad \gamma \ll \Delta^2, \tag{20}$$

where Λ is the lattice (UV) cutoff and it is assumed that $\gamma \ll \Delta^2$ and both γ and Δ are small compared to the cutoff. In the opposite case if $\gamma \gg \Delta^2$ one obtains

$$\Delta \simeq \frac{\Lambda^2}{2\sqrt{\gamma}}\exp\left\{-\frac{2\pi}{g(N-1)}\right\}, \quad \gamma \gg \Delta^2. \tag{21}$$

In two dimensions $d=2$, there is a finite second-order transition point $g=g_c$ given by

$$g_c^{-1} \simeq \frac{N-1}{\pi^2}\left(\Lambda - \frac{\pi}{2}\sqrt{\gamma}\right), \tag{22}$$

such that for $g<g_c$ the $O(N)$ symmetry is spontaneously broken and the ground state is ordered $n_0^2 = 1 - g/g_c$, while for $g>g_c$ one has a disordered phase with $n_0=0$ and $\sigma_0=\Delta^2$, where the gap Δ behaves as $\Delta \simeq \frac{4\pi}{N-1}(g_c^{-1}-g^{-1})$ at $g\to g_c$.

The transition at $g=g_c$ corresponds in our original model (1) to a transition at some critical value of anisotropic coupling $\lambda=\lambda_c$, so that one has the spin-nematic ordered phase at $\lambda>\lambda_c$ and the quantum-disordered phase at $\lambda<\lambda_c$, and the "disordered" phase actually corresponds to a dimerized state arising due to the Berry phase term.[25] The critical value λ_c can be estimated using known large-N result[25] for the critical point at $\gamma=0$ and the isotropic square lattice ($\lambda=1$),

$n_c^{\rm crit} \approx 0.19N$, which yields $\lambda_c^{-1}(\gamma=0) \approx 55.4(n_c/N)^2 - 1$. Since only $0 \le \lambda_c \le 1$ makes sense, the latter estimate suggests that in the absence of the perturbation γ the system does not order for any value of λ for $N > N_c \simeq 5$. Extrapolating to $N=3$, one obtains that at $\gamma=0$ (i.e., $\theta=-\pi/2$) the critical coupling is $\lambda_c \approx 0.19$, while the quantum Monte Carlo calculations[9] done for bilinear-biquadratic $S=1$ model yield $\lambda_c \simeq 0.13$.

B. Antiferromagnetic side ($\gamma<0$)

For $\gamma<0$ the interaction favors the minimal absolute value of z^2, so the saddle-point solution can be chosen in the form $z_0=2^{-1/2}(n_0,\ in_0,\ldots,0)$, which in the original $N=3$ model corresponds to a finite AF order parameter $\boldsymbol{l}=-i(z_0^* \times z_0)$. For the fluctuations $\boldsymbol{u}$ and $\boldsymbol{v}$ one can set $u_2 \approx 0$ and $v_1 \approx 0$ to fix the gauge, and the constraint $|z|^2=1$ yields $u_1 + v_2 \approx 0$. The mean-field equations take the form

$$n_0^2 + 2g\sum_{\boldsymbol{k}} \left\{ \frac{N-2}{\sigma_0 + k^2} + \frac{1}{\sigma_0 + 4|\gamma| + k^2} \right\} = 1,$$

$$\sigma_0 n_0 = 0. \tag{23}$$

For $d=1$ there is again only a disordered phase with $n_0=0$ and $\sigma_0=\Delta^2$, where

$$\Delta^2 \simeq \Lambda^2 \exp\left\{ -\frac{2\pi}{g(N-1)} \right\} - \frac{4|\gamma|}{N-1} \tag{24}$$

under the assumption $|\gamma| \ll \Delta^2$, and in case $\Delta^2 \ll |\gamma|$ the gap is given by

$$\Delta \simeq \Lambda \left(\frac{\Lambda^2}{4|\gamma|} \right)^{1/2(N-2)} \exp\left\{ -\frac{\pi}{g(N-2)} \right\}. \tag{25}$$

For $d=2$ the system orders at g below the critical value g_c given by

$$g_c^{-1} \simeq \frac{\Lambda(N-1)}{\pi^2} - \frac{\sqrt{|\gamma|}}{\pi}, \tag{26}$$

and for $g > g_c$ one has a disordered phase with a finite gap $\Delta=\sqrt{\sigma_0}$ which grows linearly in the vicinity of the transition $\Delta \simeq \frac{2\pi}{N-2}(g_c^{-1} - g^{-1})$.

The corresponding phase diagrams are sketched in Fig. 1. On the AF side the effect of perturbation γ is weaker by a factor of $\sim 1/N$ compared to the "nematic" case $\gamma>0$: for $d=2$ this is translated into different amplitudes of the square-root cusp in the dependence of the critical coupling λ_c on γ for $\gamma>0$ ($\theta<-\pi/2$) and $\gamma<0$ ($\theta>-\pi/2$). For $d=1$ this effect should be seen in different slopes of the gap $\Delta(\gamma)$ for positive and negative γ; this is in line with the results from exact diagonalization of small finite chains[70] as well as with the recent density-matrix renormalization-group calculations for the model (1) on a ladder.[5]

C. Influence of the perturbation on the Berry term

Up to now we have considered only the effect of the $SU(N)$-breaking perturbation γ on the action without the Berry term. Apart from favoring nematic or antiferromagnetic order, the effect consists in a mere shift of the transition point in two dimensions and a change of the gap in $d=1$ case. However, there is another important effect of the perturbation γ: as we will see, it drastically affects the Berry term, which has important consequences for the physics of the disordered phase.

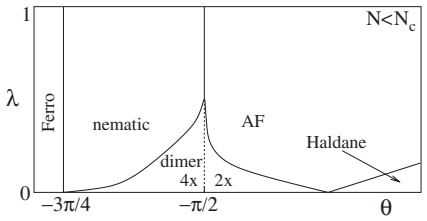

FIG. 1. A sketch of the phase diagram of the model (3) on an anisotropic square lattice in the vicinity of the $SU(N)$-symmetric point $\theta=-\pi/2$. For $N<N_c \simeq 5$ the phase boundary has a square-root-type cusp at $\theta \to -\pi/2$ as suggested by Eqs. (22) and (26). For $N>N_c$ the dimerized phase has a finite extent at $\lambda=1$. The phase denoted as "Haldane" is for $N=3$ indeed the Haldane phase whose boundary lies at $\theta=-\pi/4$, and in case of $N=4$ it is the staggered dimer phase (Refs. 46 and 49) with the boundary at $\theta=0$. For $N=3$ only, the degeneracy of the dimerized phase is twofold for $\theta > -\pi/2$ and fourfold for $\theta < -\pi/2$, see Sec. III C.

D. $d=1$

The role of the Berry phase term $\mathcal{A}_B$ at $\gamma=0$ has been studied in detail.[25,29,66] In the one-dimensional case one obtains

$$\mathcal{A}_B^{(d=1)} = i\Theta q, \quad q = \frac{1}{2\pi}\int dx d\tau F_{x\tau}, \tag{27}$$

where the integer number q has the meaning of the net topological charge (skyrmion number), and

$$\Theta = (\pi n_c \bmod 2\pi)$$

is the so-called topological angle. Explicitly expressed through z, the topological charge reads

$$q = -\frac{i}{2\pi}\int d^2x \epsilon_{\mu\nu}(\partial_\mu z^* \cdot \partial_\nu z). \tag{28}$$

For even n_c the Berry phase has no effect, while for odd n_c it leads to the twofold-degenerate ground state with a finite "static electric field" (topological charge density)

$$\langle iF_{x\tau} \rangle = \pm \frac{e_0^2}{N}. \tag{29}$$

One can easily show that the topological charge density is directly proportional to the dimerization order parameter, in essentially the same way as it has been done[71] for the $O(3)$ nonlinear sigma model. Indeed, the dimerization operator at $\gamma=0$ can be defined as

$$\mathcal{O}_n^{\rm dim} = \eta_n (\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1})^2 \tag{30}$$

and after passing to the continuum its leading nonoscillating part will take the form

$$\mathcal{O}_n^{\mathrm{dim}} \mapsto (\boldsymbol{\psi}_n \cdot \boldsymbol{z}_{n+1}^* - \boldsymbol{z}_n \cdot \boldsymbol{\psi}_{n+1}^*) + \text{c.c.}$$
$$\mapsto 2(\boldsymbol{\psi} \cdot \partial_x \boldsymbol{z}^* + \boldsymbol{\psi}^* \cdot \partial_x \boldsymbol{z}) \mapsto [2\tilde{J} n_c (1+\lambda)]^{-1} (\partial_\tau \boldsymbol{z}^* \cdot \partial_x \boldsymbol{z} - \partial_\tau \boldsymbol{z} \cdot \partial_x \boldsymbol{z}^*)$$
$$= [2\tilde{J} n_c (1+\lambda)]^{-1} (i F_{x\tau}) \tag{31}$$

Thus, at the $SU(N)$-symmetric point $\gamma=0$ the ground state for odd n_c and $N \geq 3$ is spontaneously dimerized.[25] The case $N=2$, however, is an exception: for $N=2$ the model is equivalent to the $O(3)$ nonlinear sigma model with the topological angle $\Theta=\pi$, which is gapless and nondimerized[66–68] in a wide range of the coupling g.

Let us first illustrate the effect of the $SU(N) \mapsto O(N)$ perturbation γ on the Berry phase by a simple observation[57] valid for $N=3$. Finite $\gamma<0$ favors field configurations of the antiferromagnetic type, namely, $\boldsymbol{z}=\frac{1}{\sqrt{2}}(\boldsymbol{e}_1+i\boldsymbol{e}_2)$, with $\boldsymbol{e}_{1,2}$ being two orthonormal vectors and $\boldsymbol{n}(\theta,\varphi)=\boldsymbol{e}_1 \times \boldsymbol{e}_2$ having the meaning of the unit Néel vector characterized by two spherical angles θ and φ. It is a straightforward exercise to check that

$$q = \frac{1}{2\pi} \int d^2x \, \sin\theta \epsilon_{\mu\nu} (\partial_\mu \theta)(\partial_\nu \varphi)$$
$$= \frac{1}{4\pi} \int d^2x \epsilon_{\mu\nu} \boldsymbol{n} \cdot (\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n}) = 2Q, \tag{32}$$

where the topological charge Q is the winding number of the $S^2 \mapsto S^2$ mapping characterizing the space-time distribution of the unit vector $\boldsymbol{n}(\theta,\varphi)$. This shows that negative γ favors z-field configurations with *even* charge q and suggests that configurations with odd q become suppressed. This is physically important because if odd-q configurations are prohibited, the Berry term obviously becomes ineffective, irrespectively of whether n_c is even or odd. The above argument cannot be applied for $N>3$ because the second homotopy group of $O(N>3)$ sigma models is trivial so they possess no π_2 topological charge. It is also not possible to extend this argument to $\gamma>0$ because in this case "nematic" configurations with z being a real (up to an arbitrary overall phase) unit vector are favored and for such configurations the CP^{N-1} topological charge (28) identically vanishes.

To understand what happens in case of general N and γ, consider the general one-skyrmion ($q=1$) solution of the 1+1-dimensional CP^{N-1} model which has the form

$$z_\alpha = \frac{c_\alpha (Z - a_\alpha)}{\left(\sum_\beta |c_\beta|^2 |Z - a_\beta|^2 \right)^{1/2}}, \tag{33}$$

where $Z=x_0+ix_1$ is the complex coordinate, the complex numbers a_α have the meaning of coordinates of the N skyrmion constituents (sometimes called "zindons" from a Persian word meaning prison[72]) and another set of complex numbers c_α may be viewed as amplitudes associated with each zindon. Normalizing the amplitudes c_α as $\Sigma_\alpha |c_\alpha|^2=1$, putting the origin into the "center of mass" (which amounts to demanding $\Sigma_\alpha |c_\alpha|^2 a_\alpha=0$), and defining the average "size" R of the skyrmion as the dispersion of the zindon positions,

$$R^2 \equiv \sum_\alpha |c_\alpha|^2 |a_\alpha|^2, \tag{34}$$

one can recast the general $q=1$ solution (33) in a more elegant form[63]

$$\boldsymbol{z} = \frac{\boldsymbol{U}R + \boldsymbol{V}Z}{(|Z|^2 + R^2)^{1/2}}, \tag{35}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are two orthonormal complex N-component vectors,

$$\boldsymbol{U}^* \cdot \boldsymbol{U} = \boldsymbol{V}^* \cdot \boldsymbol{V} = 1, \quad \boldsymbol{U}^* \cdot \boldsymbol{V} = 0. \tag{36}$$

For $\gamma=0$, i.e., in the unperturbed CP^{N-1} model, the action of such skyrmion solution does not depend on its parameters. For a finite γ, however, one gets an additional contribution to the action from the $|z^2|^2$ term (17).

Let us calculate this correction to the first order in γ. Consider first the "antiferromagnetic" case $\gamma<0$. To minimize the action cost, we must reduce as much as possible the deviations of z^2 from zero. Requiring $\boldsymbol{U}^2=0$ ensures that $z^2 \to 0$ at $|Z| \to \infty$, killing the next leading term in Z fixes $\boldsymbol{U} \cdot \boldsymbol{V}=0$, and, finally, if we were able to satisfy additionally $\boldsymbol{V}^2=0$ then the condition $z^2=0$ would be identically fulfilled. Those three constraints can be satisfied together with Eq. (36) only if the four real N-component vectors $\mathrm{Re}(\boldsymbol{U})$, $\mathrm{Re}(\boldsymbol{V})$, $\mathrm{Im}(\boldsymbol{U})$, and $\mathrm{Im}(\boldsymbol{V})$ are mutually orthogonal, which is readily achieved for $N \geq 4$ but is obviously impossible for $N=3$. Thus, *for* $N \geq 4$ the $q=1$ skyrmion (35) remains an exact solution even for finite $\gamma<0$. In other words, when $\gamma<0$ is switched on, the "zindons" constituting a skyrmion are able to adjust themselves for $N \geq 4$ in such a way that the skyrmion continues to provide the minimum of action. This is in fact amusing because formally for $\gamma<0$ the model has only the $O(N)$ symmetry and one would expect that skyrmions do not exist for $N>3$.

For $N=3$ and $\gamma<0$ the minimum contribution of the perturbation to the action of the skyrmion (35) is achieved if $\boldsymbol{V}$ is real and the three vectors $\boldsymbol{V}$, $\mathrm{Re}(\boldsymbol{U})$, and $\mathrm{Im}(\boldsymbol{U})$ are mutually orthogonal. The excess action due to finite γ is then given by

$$\Delta \mathcal{A}_{\gamma<0} = -\frac{\gamma}{2g} \int d^2x |z^2|^2 = -\frac{\pi \gamma R^2}{2g}, \tag{37}$$

and it grows as a square of the skyrmion size which means at $\gamma<0$ the field configurations with $q=1$ are prone to collapse and only exist as metastable "excitations." At the same time, one can easily adjust the parameters of general $q=2$ skyrmion solutions of the $\gamma=0$ model so that $z^2=0$ is identically satisfied (see the Appendix). This effect can be interpreted as "topological pairing" of $q=1$ skyrmions.[57] The total topological charge density $F_{x\tau}$ can be separated into two parts $F_{x\tau}^{(q=1)}$ and $F_{x\tau}^{(q=2)}$ which correspond to the contributions from unbound $q=1$ skyrmions and their bound pairs, respectively. Only $F_{xt}^{(q=1)}$ contributes to the nontrivial part of the Berry phase (27), while the full $F_{x\tau}$ enters the gauge-field kinetic-energy term (16). The dimerization order parameter $\langle \mathcal{O}^{\mathrm{dim}} \rangle \propto \langle i F_{x\tau} \rangle$ will be proportional to the fraction ρ of the $q=1$ skyrmions and so will be diminishing with increasing $|\gamma|$.

This "topological" suppression of dimerization at $\gamma<0$ exists only for $N=3$ and is absent for $N\geq 4$, which implies that within our description the dimerized phase for $N\geq 4$ should extend to the entire region $\gamma<0$ [i.e., up to $\theta=\pi/4$ which in our notation is another $SU(N)$-symmetric point of the model, corresponding to the transition into a critical phase[35]]; however, it is clear that our description will eventually break down as the "perturbation" $|\gamma|$ becomes large. In fact, according to exact results (see Ref. 73) for one-dimensional $SO(N)$ generalizations of the bilinear-biquadratic model (1), even for $N\geq 4$ there is still a phase transition on the way from the AF $SU(N)$ point to the critical $SU(N)$ one. The chunk of different phase lying between the dimerized phase and the critical $SU(N)$ point gets squeezed with increasing N, and the transition point for $N\geq 4$ lies in the region of $\gamma\gtrsim 1$, which is way beyond the range of applicability of the present approach.

Consider now the perturbation of the opposite sign $\gamma>0$, which favors nematiclike field configurations with $z=e^{i\alpha}\boldsymbol{\varphi}$, where α is an arbitrary phase and $\boldsymbol{\varphi}$ is a real unit vector. For such z the topological charge (28) is identically zero, which indicates that skyrmions with any charge are suppressed by the perturbation. In a different way one can see that by calculating the γ-dependent correction to the action. For a $q=1$ skyrmion (35) minimizing the deviation of $|z^2|$ from one leads to the requirement that $\boldsymbol{U}$ and $\boldsymbol{V}$ are real, and the resulting correction diverges logarithmically with the system size L,

$$\Delta\mathcal{A}_{\gamma>0}\simeq\frac{2\pi\gamma R^2}{g}\ln\frac{L}{R}. \tag{38}$$

In the disordered phase, one expects that the system size L above will be replaced by the correlation length ξ. In contrast to the AF-like case $\gamma<0$, this suppression persists for any number of the field components N. A similar calculation for $q=2$ yields $\Delta\mathcal{A}_{\gamma>0}^{q=2}\propto\gamma R^2$, so the even-charged skyrmions are suppressed as well though weaker than the odd-charged ones. Thus, with increasing γ the contribution from smooth field configurations (skyrmions) to the Berry phase dies out, but at the same time the contribution from discontinuous configurations (disclinations or Z_2 vortices) remains unaffected and gradually becomes the leading one. Indeed, a configuration with a real vector z abruptly changing sign across some bond along a path running in the time direction contributes the Berry phase equal to π for every such bond,[10,22] which is not captured by the continuum-limit expression (27) but is readily seen from the general formula (10). In the disordered phase the fluctuations of z are gapped and can be integrated out, leaving one only with Ising-type degrees of freedom marking bonds where a discontinuous change $z\rightarrow -z$ occurred.[10] The resulting so-called odd Z_2 gauge theory[74] is known to be always dimerized in one dimension,[75] which, according to Grover and Senthil,[10] explains why the dimerized phase extends all the way up to $\theta=-3\pi/4$ (their arguments can be literally transferred to the effective theory of Ref. 58 which is suited for describing the region $-3\pi/4<\theta\lesssim -0.65\pi$ with ferro-type local correlations).

E. $d=2$

In two dimensions the Berry phase is determined by instanton processes ("monopoles") changing the skyrmion topological quantum number q and is given by[25,60]

$$\mathcal{A}_B=\frac{i\pi n_c}{2}\sum_{\boldsymbol{r}_i}\zeta(\boldsymbol{r}_i)\tilde{q}_i, \tag{39}$$

where the sum is over the locations $\boldsymbol{r}_i$ of monopoles having the charge $\tilde{q}_i$ (i.e., the skyrmion number gets changed by $\tilde{q}_i$), and the factor $\zeta(\boldsymbol{r}_i)$ takes values of 0, 1, 2, and 3 for $\boldsymbol{r}_i$ belonging to the four dual sublattices W, X, Y, and Z, respectively (see Fig. 7 of Ref. 25). At the $SU(N)$-symmetric point $\gamma=0$ for $n_c\neq 0 \bmod 4$ the Berry term leads to the ground state with nonzero instanton density, thus to finite electric fields and to spontaneous breaking of translation symmetry:[25] the dimerized ground state is twofold degenerate for $n_c=2 \bmod 4$ and fourfold degenerate for $n_c=(1 \text{ or } 3)\bmod 4$.

When the $SU(N)$-breaking perturbation γ is switched on, the monopoles are transformed in a similar way as in $d=1$ case for skyrmions: at $\gamma<0$ monopoles with odd $\tilde{q}$ are strongly suppressed for $N=3$ and remain unaffected for $N\geq 4$. Suppression of odd-charged monopoles for $N=3$ and $\gamma<0$ can be understood by invoking the same type of argument as that we have used in the one-dimensional case. The monopole charge $\tilde{q}=\oint j_\alpha dS_\alpha$ can be defined as the quantized flux of the "skyrmion current"

$$j_\alpha=\frac{1}{2\pi}\varepsilon_{\alpha\mu\nu}\frac{\partial A_\nu}{\partial x_\mu}=-\frac{i}{2\pi}\varepsilon_{\alpha\mu\nu}(\partial_\mu z^*\cdot\partial_\nu z) \tag{40}$$

through a closed surface surrounding the monopole. For antiferromagnetic-type configurations favored at $\gamma<0$ one again can write $z=\frac{1}{\sqrt{2}}(\boldsymbol{e}_1+i\boldsymbol{e}_2)$, where $\boldsymbol{e}_{1,2}$ are two orthonormal vectors, and define the corresponding $O(3)$ unit unit vector field $\boldsymbol{l}=\boldsymbol{e}_1\times\boldsymbol{e}_2$. Then it is easy to obtain

$$\begin{aligned} j_\alpha&=\frac{1}{2\pi}\varepsilon_{\alpha\mu\nu}(\partial_\mu\boldsymbol{e}_2\cdot\partial_\nu\boldsymbol{e}_1)\\ &=\frac{1}{2\pi}\varepsilon_{\alpha\mu\nu}[\boldsymbol{e}_1\cdot(\boldsymbol{e}_2\times\partial_\mu\boldsymbol{e}_2)][\boldsymbol{e}_2\cdot(\boldsymbol{e}_1\times\partial_\nu\boldsymbol{e}_1)]\\ &=\frac{1}{4\pi}\varepsilon_{\alpha\mu\nu}\boldsymbol{l}\cdot(\partial_\mu\boldsymbol{l}\times\partial_\nu\boldsymbol{l})\equiv 2J_\alpha, \end{aligned} \tag{41}$$

where J_μ is the corresponding skyrmion current of the $O(3)$ nonlinear sigma model whose flux through a closed surface should be an integer number. Again, this argument only works for $N=3$. For $N\geq 4$ the $\tilde{q}=1$ monopole solution[76]

$$z=\boldsymbol{U}\cos(\theta/2)e^{i\varphi}+\boldsymbol{V}\sin(\theta/2), \tag{42}$$

where θ and φ are the angular spherical coordinates in the $(2+1)$-dimensional space, and the monopole is assumed to be placed at the origin, can be easily adjusted to yield $z^2=0$ identically for $N\geq 4$, and for $N=3$ the excess action due to the perturbation γ of such a monopole diverges as the space-time volume (note that this contribution arises due to deviation of z^2 from zero and thus is not destroyed by the vanishing spin stiffness in the disordered phase as the con-

tribution from the main $\mathcal{A}_0$ part of the action does[76]). Even-charged monopoles can be shown to survive for a finite $\gamma<0$ as exact solutions (see the Appendix). So, we come to the conclusion that *at* $\gamma<0$ the odd-charged monopoles get confined into pairs for $N=3$ but are insensitive to the perturbation for $N\geq 4$. The consequence for $N=3$ is that the contribution of odd-charged monopoles is switched off for any finite $\gamma<0$, which effectively amounts to doubling n_c in Eq. (39); for the bilinear-biquadratic $S=1$ model (1) that means that the dimerized state is doubly degenerate at $\gamma<0$ and becomes fourfold degenerate only at $\gamma=0$.

On the nematic side ($\gamma>0$) the effective theory has been constructed by Grover and Senthil;[10] they have shown that the problem can be mapped to an XY model with a fourfold anisotropy term. The dimerized ground state is, respectively, predicted to be fourfold degenerate in that case. One is thus led to conclude that $\theta=-\pi/2$ for $N=3$ should be the first-order transition line.

IV. EFFECT OF THE $SU(N)\mapsto SU(N-1)$ PERTURBATION

Consider now a different way to perturb the $SU(N)$ symmetry, namely, let us introduce a finite mass for one of the components of the z field,

$$\mathcal{A}\mapsto\mathcal{A}+\frac{m_0^2}{2g}\int d^{d+1}x|z_N|^2, \tag{43}$$

which for the $S=1$ model (1) is equivalent to including the *easy-axis* single-ion anisotropy term (5) with $D=2gm_0^2$. For cold atoms in optical lattices, such terms appear naturally in the presence of external magnetic field due to the quadratic Zeeman coupling.[23,56] This perturbation breaks the $SU(N)$ symmetry of the model down to $SU(N-1)$ and produces a CP^{N-2} model with the topological angle $\Theta=\pi$ as the effective theory. Actually, the operators $N_\pi^{zz}=\Sigma_{\boldsymbol{n}}\eta_{\boldsymbol{n}}t_{\boldsymbol{n},N}^\dagger t_{\boldsymbol{n},N}$ and $N_0^{zz}=\Sigma_{\boldsymbol{n}}t_{\boldsymbol{n},N}^\dagger t_{\boldsymbol{n},N}$ commute with the Hamiltonian (3) if $\cos\theta=0$ and $\sin\theta=0$, respectively. So, at the "ferro-$SU(3)$" point $\theta=-5\pi/4$ the single-ion anisotropy D acts simply as an "external field" coupling to a conserved quantity, but at the "AF-$SU(3)$" point $\theta=-\pi/2$ the situation is different.

In one dimension ($d=1$), if the mass m_0 is large compared to the gap $\Delta\simeq\Lambda\exp\{-\pi/g(N-1)\}$, one can integrate out just the single most massive Nth component and obtain in that way a correspondence between the *bare* coupling constant g_{N-1} of the effective CP^{N-2} model and the bare coupling constant $g\equiv g_N$ of the original CP^{N-1} model,

$$g_{N-1}=\frac{g}{1-\frac{g}{2\pi}\ln\left(1+\frac{\Lambda^2}{m_0^2}\right)}. \tag{44}$$

Now, the case $N=3$ is again exceptional because the CP^1 model with the topological angle $\Theta=\pi$ in $d=1$ is gapless in an extended range of coupling. At infinite coupling $g_2=\infty$ the parity is broken[77] and several approaches[66,67] indicate that there is a parity-breaking dimerization transition at a strong but finite value of g_2, although it seems the answer may depend on the specific lattice realization.[68] Thus, at least for some range of $g\equiv g_3$ the coupling g_2 will flow to zero and one expects a phase transition for $N=3$ on the way from $m_0=0$ to ∞. For $N\geq 4$ the resulting CP^{N-2} model with $\Theta=\pi$ remains dimerized, so no phase transition takes place.

In the two-dimensional case, a usual poor-man's renormalization group calculation yields the effective coupling $\tilde{g}$ as a function of the anisotropy m_0^2,

$$\tilde{g}=\frac{g}{1-\frac{g}{2\pi^2}\left[(N-1)\Lambda-m_0\arctan\frac{\Lambda}{m_0}\right]}. \tag{45}$$

Depending on the value of the bare coupling $g=n_c^{-1}\sqrt{1+1/\lambda}$, there are three possible scenarios: (a) if $g<g_N^{(c)}=\pi^2/[\Lambda(N-1)]$ then the system has long-range nematic or AF order all the way from $m_0=0$ to ∞; (b) if $g_N^{(c)}<g<g_{N-1}^{(c)}$ then the system is disordered (and dimerized) at $m_0=0$, but with increasing m_0 there is an ordering transition at $m_0\simeq(2\pi/g_0)(1-g/g_N^{(c)})$; finally, if $g>g_{N-1}^{(c)}$, the system stays dimerized at all values of m_0. The combined effect of the $SU(N)\mapsto SU(N-1)$-breaking perturbation (43) and the $SU(N)\mapsto O(N)$ one [Eq. (17)] is also transparent: taken together, those terms lower the symmetry to $O(N-1)$, and for $N\geq 4$ the corresponding behavior as a function of γ at finite m_0 can be inferred from the behavior of the model with $N\to N-1$. In one dimension, for $N=3$ and at large m_0, $\gamma>0$ favors a phase with dominant power-law XY-type nematic correlations (the $XY2$ phase in the classification of Schulz[78]), while $\gamma<0$ favors the Ising-type long-range antiferromagnetic order. The transition from the XY nematic to the dimerized phase is of the Berezinskii-Kosterlitz-Thouless type and the transition from the AF Ising to the dimerized phase belongs to the Ising universality class. The corresponding phase diagrams are sketched in Fig. 2.

For the spin-1 model (1) that corresponds to $N=3$, it is instructive to construct the effective Hamiltonian in the limit of strong single-ion anisotropy $D\gg 1$. Indeed, in that limit the Hilbert space at each site $\boldsymbol{n}$ is effectively reduced to the two spin-1 states $|+\rangle$ and $|-\rangle$, which can be identified with $|\uparrow\rangle$ and $|\downarrow\rangle$ states of pseudospin-$\frac{1}{2}$. In the second order of perturbation theory in $1/D$, the effective Hamiltonian is given by the XXZ model in terms of pseudospin-$\frac{1}{2}$ operators $\boldsymbol{\tau}_{\boldsymbol{n}}$

$$\mathcal{H}_{\text{eff}}=\sum_{\boldsymbol{n}}\{\tilde{h}_{\boldsymbol{n},\boldsymbol{n}+\boldsymbol{x}}+\lambda\tilde{h}_{\boldsymbol{n},\boldsymbol{n}+\boldsymbol{y}}\},$$

$$\tilde{h}_{\boldsymbol{n},\boldsymbol{n}'}=-\tilde{J}_{xy}(\tau_{\boldsymbol{n}}^x\tau_{\boldsymbol{n}'}^x+\tau_{\boldsymbol{n}}^y\tau_{\boldsymbol{n}'}^y)+\tilde{J}_z\tau_{\boldsymbol{n}}^z\tau_{\boldsymbol{n}'}^z,$$

$$\tilde{J}_{xy}\simeq-2\sin\theta+\frac{(\cos\theta-\sin\theta)^2}{D},$$

$$\tilde{J}_z=\tilde{J}_{xy}+4\cos\theta. \tag{46}$$

One can see that for $\theta=-\pi/2$ the effective Hamiltonian is $SU(2)$ symmetric, in agreement with the continuum field description. Deviations from $\theta=-\pi/2$ break this $SU(2)$ symmetry, favoring AF or nematic order.

V. SUMMARY

We have studied the consequences of explicit symmetry breaking in the model of low-dimensional $SU(N)$ antiferro-

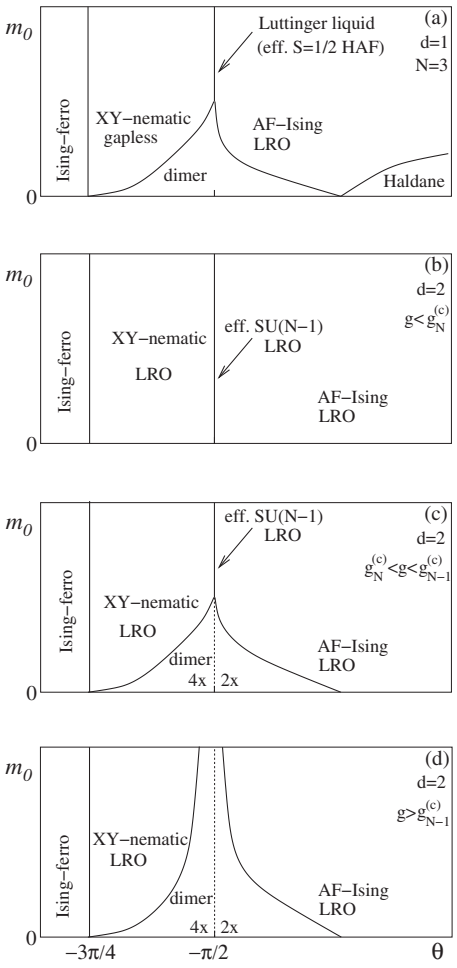

FIG. 2. A sketch of the phase diagram of the model (3) in the vicinity of the $SU(N)$-symmetric point $\theta=-\pi/2$, in the presence of two symmetry-breaking perturbations (17) and (43): (a) the one-dimensional case with $N=3$; for $N\geq 4$ only the dimerized phase survives around the $SU(N)$-symmetrical point; (b)–(d) the two-dimensional case for different values of the bare coupling $g=n_c^{-1}\sqrt{1+\lambda^{-1}}$; here $g_N^{(c)}=\pi^2/[\Lambda(N-1)]$; for $N=3$ only, the degeneracy of the dimerized phase changes from fourfold at $\theta<-\pi/2$ to twofold at $\theta>-\pi/2$.

magnet on a bipartite lattice motivated by the physics of cold spinor bosonic atoms in optical lattices. Two possible routes have been considered: lowering the $SU(N)$ symmetry down to $O(N)$ and to $SU(N-1)$. Physically, in cold atom systems those perturbations naturally arise due to the presence of the external magnetic field which controls the detuning from the Feshbach resonance and simultaneously causes the quadratic Zeeman effect. Both ways of the symmetry breaking result in rich sequences of transitions between dimerized, antiferromagnetic, and spin-nematic phases. The qualitative form of the phase diagram depending on the model parameters is established. It is shown that the physically interesting case $N=3$ is special: perturbing the $SU(3)$ symmetry leads to nontrivial changes in the Berry phases, which are reflected in the degeneracy of the dimerized phase.

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APPENDIX: EVEN-CHARGED SKYRMIONS AND MONOPOLES IN THE PERTURBED CP^2 MODEL

Consider a general $q=2$ skyrmion solution of the unperturbed ($\gamma=0$) (1+1)-dimensional CP^2 model, which has the form

$$z_\alpha = f_\alpha/|f|, \quad f_\alpha = a_\alpha Z^2 + b_\alpha Z + c_\alpha. \tag{A1}$$

Now we would like to adjust the parameters of the above solution to satisfy $z^2=0$, making it suitable for $\gamma<0$. Denoting the real and imaginary parts of the three-component complex vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ as $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, etc., we see that ($\boldsymbol{a}_1$, $\boldsymbol{a}_2$) must be (up to a scale factor) a pair of mutually orthogonal unit vectors and the same is true for ($\boldsymbol{c}_1$, $\boldsymbol{c}_2$). We choose the coordinate system so that $\boldsymbol{c}_1\|\hat{\boldsymbol{x}}$ and $\boldsymbol{c}_2\|\hat{\boldsymbol{y}}$ and set $b_z=2R>0$ to fix the overall phase and norm. The following solution does the job:

$$\boldsymbol{b}=\{\zeta^*,\zeta,2R\}, \quad \boldsymbol{c}=R^2\{1,i,0\}, \quad \boldsymbol{a}=\boldsymbol{a}_1+i\boldsymbol{a}_2,$$

$$\boldsymbol{a}_1=\{-1-\lambda^2\cos 2\chi,\lambda^2\sin 2\chi,2\lambda\sin\chi\},$$

$$\boldsymbol{a}_2=\{-\lambda^2\sin 2\chi,1-\lambda^2\cos 2\chi,-2\lambda\cos\chi\}, \tag{A2}$$

where ζ is an arbitrary complex number and λ and χ are real. It is easy to convince oneself that this solution is nothing but the disguised Belavin-Polyakov skyrmion[79] of the $O(3)$ nonlinear sigma model with the topological charge $Q=1$. The correspondence between the CP^2 field $\boldsymbol{z}$ and the sigma-model unit vector $\boldsymbol{l}$ is given by $\boldsymbol{l}=-i(\boldsymbol{z}^*\times\boldsymbol{z})$, and the $O(3)$ topological charge is determined by Eq. (32). One can easily see that the simplest Belavin-Polyakov solution $(\ell_1+i\ell_2)/(1-\ell_3)=Z/R$ translates into

$$z=\frac{1}{\sqrt{2}(|Z|^2+R^2)}\{Z^2+R^2,i(R^2-Z^2),2iRZ\}$$

which after a rotation $Z\mapsto Ze^{-i\pi/2}$ becomes a special case of Eq. (A2) with $\lambda=0$, $\chi=\pi/2$, and $\zeta=0$. This solution describes a $q=2$ skyrmion whose six constituents ("zindons") sit at $Z=\pm R$, $Z=\pm iR$, $Z=0$, and $Z=\infty$.

In a similar way, one can show that in (2+1) dimensions a monopole of the CP^2 model with the even integer charge $\tilde{q}=2m$ defined as a solution to the equation[80]

$$\varepsilon_{\alpha\mu\nu}\frac{\partial A_\mu}{\partial x_\nu}=\frac{\tilde{q}x_\mu}{2r^3}, \tag{A3}$$

where $r^2=\Sigma_\mu x_\mu^2$ and the monopole is assumed to be at the origin, corresponds exactly to the hedgehog solution of the $O(3)$ model with a charge $\tilde{Q}=m$. Indeed, it is straightforward to check that the solution of the form

$$z = \frac{1}{\sqrt{2}} \begin{Bmatrix} \cos\,\theta \cos(m\varphi) - i \sin(m\varphi) \\ \cos\,\theta \sin(m\varphi) + i \cos(m\varphi) \\ -\sin\,\theta \end{Bmatrix}, \tag{A4}$$

where θ and φ are the spherical angular coordinates in the $(2+1)$-dimensional space, satisfies Eq. (A3) with $\tilde{q}=2m$, satisfies $z^2=0$, and its corresponding $O(3)$ unit unit vector field $\boldsymbol{l}=-i(z^*\times z)$ describes a $\tilde{Q}=m$ hedgehog,

$$\boldsymbol{l} = \{\sin\,\theta\cos\,\varphi, \sin\,\theta\sin\,\varphi, \cos\,\theta\}. \tag{A5}$$

This confirms that the even-charged monopoles (A4) remain exact solutions even in the perturbed case (but only for $\gamma < 0$). Odd-charged monopoles are suppressed as explained in Sec. III C.

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